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Analysis of turbulent pipe and channel flows at moderately large Reynolds number

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Effects of moderately large Reynolds numbers R are studied by considering higher order terms in the expansions for turbulent pipe and channel flows for $R \to \infty$. Matched asymptotic expansions using two length scales are employed to emphasize the two-layer structure of turbulent shear flows near solid walls. The effects appear as additional terms in extended forms of the law of the wall, the logarithmic velocity law, the velocity defect law and the logarithmic skin-friction law. These generalizations are critically compared with experimental results for pipe flows of Patel & Head and extremely good agreement is obtained. Also, possible applications are discussed for extending the range of skin-friction and heat-transfer devices which are based on wall similarity.

1. Introduction

Properties of turbulent shear flows at moderately large Reynolds numbers† have attracted the attention of several workers in recent years. For instance, Patel & Head (1969) reported new observations on turbulent pipe and channel flows in the Reynolds number range (based on average velocity and pipe diameter or channel depth) 1–10^4. Huffman & Bradshaw (1972) have recently performed calculations on the basis of a closure hypothesis for a similar range of Reynolds numbers.

The aim of this paper is to study some of these properties by using asymptotic expansions for large Reynolds numbers. Higher order terms in the expansions describe Reynolds number effects which may be detected experimentally when the Reynolds number is not too large.

Asymptotic expansions for turbulent shear flows have of late been increasingly studied (Gill 1968; Tennekes 1968; Yajnik 1970; Afzal & Yajnik 1971; Afzal 1971, 1973; Chawla & Tennekes 1973; Mellor 1972). Higher order effects in pipe flow were first studied by Tennekes (1968) using a hypothetical analogy between

† The phrase moderately large Reynolds number is used here to denote the range beyond completion of transition in which some Reynolds number effects in classical correlations have been found.

these higher order effects in pipe flow and the first-order effects in the zero-skin-friction turbulent boundary layer (Tennekes 1966; see also Chawla & Tennekes 1973).

The analysis given here is free from the customary uncertainty arising from closure hypotheses, since it deals with an underdetermined system of equations. Since functional forms describing qualitative properties of the flows can be obtained by limiting and matching arguments and by integration processes, it is not necessary to make the system fully determined. However, the determination of a unique flow for given boundary conditions using limiting equations requires additional information directly or indirectly describing the statistical mechanics of turbulent fluctuations.

While the method in its essential content is general, attention is focused on fully developed pipe and channel flows, for which extensive data exist. Higher order effects in boundary layers are discussed by Afzal (1971, 1973) and Mellor (1972).

2. Analysis

Consider the fully developed turbulent flow of a constant-property Newtonian fluid in a hydraulically smooth pipe or channel. Here the fully developed state refers to invariance of the mean velocity, pressure gradient and Reynolds stresses in the streamwise direction, but the flow need not be fully developed in any of the three senses discussed by Patel & Head (1969).

It is convenient to start from an integral of the mean motion† (see, for example, Hinze 1959) written as

$$\epsilon U_* U' - \tau = U_*^2(1-2y). \tag{1}$$

Here all variables are in non-dimensional form with the pipe diameter or channel depth and the cross-sectional average of velocity as the reference length and velocity. y is the normal co-ordinate measured from the wall and U and τ are the mean velocity and Reynolds stress. ϵ is defined as $(U_* R)^{-1}$, where U_* is the dimensionless friction velocity and R is the Reynolds number based on the reference length and velocity. A prime denotes differentiation with respect to y.

Conditions at the wall require that U, τ, τ' and τ'' vanish there and additional conditions at the pipe axis or channel plane of symmetry may be obtained by assuming symmetry of the mean flow.

We seek two limits and two corresponding asymptotic expansions which describe the flow region close to the wall and the flow region away from it. The length scales of these regions are of order ϵ and unity. However, the corresponding scales for the inner and outer parts of a turbulent boundary layer are of order ϵ and U_*, and their ratio is $\hat{\epsilon} = (U_*^2 R)^{-1}$, while in pipe and channel flows the ratio is $\epsilon = (U_* R)^{-1}$. Hence the appropriate expansion parameter for pipe and channel flows turns out to be ϵ and not $\hat{\epsilon}$ as pointed out by Mellor (1972).

We introduce outer variables $U_0 = (U-1)/U_*$ and $\tau_0 = \tau/U_*^2$ and inner

† The integral in the above form is applicable to both pipe and channel flows. When the integrals are written in terms of the pressure gradient, one term differs by a factor of $\frac{1}{2}$.

variables $u_i = U/U_*$, $\tau_i = \tau/U_*^2$ and $\eta = y/\epsilon$. Equation (1) can be then written in these variables as

$$\epsilon\, dU_0/dy - \tau_0 = 1 - 2y \tag{2}$$

and

$$du_i/d\eta - \tau_i = 1 - 2\epsilon\eta. \tag{3}$$

These equations contain only ϵ and are in a very simple form. One may hope that expansions are in integral powers of ϵ. Let the outer and inner expansions be

$$U_0 = U_1(y) + \epsilon U_2(y) + \epsilon^2 U_3(y) + \ldots, \tag{4a}$$

$$\tau_0 = T_1(y) + \epsilon T_2(y) + \epsilon^2 T_3(y) + \ldots; \tag{4b}$$

$$u_i = u_1(\eta) + \epsilon u_2(\eta) + \epsilon^2 u_3(\eta) + \ldots, \tag{5a}$$

$$\tau_i = \tau_1(\eta) + \epsilon \tau_2(\eta) + \epsilon^2 \tau_3(\eta) + \ldots. \tag{5b}$$

Here the outer and inner limits are for fixed y and fixed η as $\epsilon \to 0$, $1/\epsilon$ being the Reynolds number based on the friction velocity.

It then follows from (2) and (3) that the outer and inner equations are

$$T_1 = 2y - 1, \tag{6a}$$

$$dU_n/dy = T_n \quad (m = 1, 2); \tag{6b}$$

$$du_1/d\eta = \tau_1 + 1, \quad du_2/d\eta = \tau_2 - 2\eta. \tag{7a,b}$$

Matching

There are several ways of matching inner and outer expansions. We shall use here the matching condition given by Van Dyke (1964, p. 90):

$$\mathscr{I}_m \mathscr{O}_n(f) = \mathscr{O}_n \mathscr{I}_m(f), \tag{8}$$

where $\mathscr{I}_m(f)$ and $\mathscr{O}_m(f)$ denote the m-term inner and outer expansions of the function f.

When we match the Reynolds stress in the inner and outer region for $m = 1, 2$ and $n = 1$ we find that $\tau_1 \sim -1$ and $\tau_2 \sim 2\eta$ as $\eta \to \infty$. The simple forms of (6) and (7) suggest that expansions of τ_1 and τ_2 in integral powers of η may be adequate. Such expansions are consistent with the assumption of the expansions (4) and (5) in integral powers of ϵ. We then assume that

$$\left.\begin{aligned} \tau_1 &\sim -1 + b_1/\eta + c_1/\eta^2 + d_1/\eta^3 + \ldots & (9a)\\ \tau_2 &\sim 2\eta + a_2 + b_2/\eta + c_2/\eta^2 + d_2/\eta^3 + \ldots & (9b) \end{aligned}\right\} \quad \text{as} \quad \eta \to \infty.$$

Now the two-term inner expansion of τ may be expressed in the outer variable y using (9a) and (9b):

$$\mathscr{I}_2(\tau) = U_*^2[2y - 1 + \epsilon(b_1/y + a_2) + \epsilon^2(c_1/y^2 + b_2/y + a_3)]. \tag{10}$$

Here a_3 is a constant term in the expansion for τ_3. The matching conditions for $m = 2$ and $n = 2, 3$ give

$$\left.\begin{aligned} T_2 &\sim b_1/y + a_2 + \ldots & (11a)\\ T_3 &\sim c_1/y^2 + b^2/y + a_3 + \ldots & (11b) \end{aligned}\right\} \quad \text{as} \quad y \to 0.$$

Equations (6) and (7) lead to expansions of dU_n/dy and $du_n/d\eta$ in integral powers of y and η. We find that the expansions for dU/dy in the overlap region are given by

$$dU_o/dy \sim (b_1/y+a_2+\ldots)+\epsilon(c_1/y^2+b_2/y+a_3+\ldots)+\ldots \quad \text{as} \quad y \to 0, \tag{12a}$$

$$du_i/d\eta \sim (b_1/\eta+c_1/\eta^2+\ldots)+\epsilon(a_2+b_2/\eta+c_2/\eta^2+\ldots)+\ldots \quad \text{as} \quad n \to \infty, \tag{12b}$$

and they satisfy the matching conditions (8).

Explicit forms for the wall region can be obtained from integration of (7) by using the matching and boundary conditions:

$$u_1 = b_1 \ln(1+\eta)+\lambda_1(\eta), \tag{13a}$$

$$u_2 = a_2\eta+b_2 \ln(1+\eta)+\lambda_2(\eta), \tag{13b}$$

where

$$\lambda_n(\eta) = \int_0^\eta [\tau_n - 2\eta\delta_{1n}+\delta_{2n}+b_n/(1+\eta)]\,d\eta \quad (n = 1, 2). \tag{14}$$

The integrals are bounded as $\eta \to \infty$ and δ_{ij} is the Kronecker delta.

Similarly, explicit forms for the core region can be obtained from (6) as

$$U_1 = b_1 \ln y+\Lambda_1(y), \tag{15a}$$

$$U_2 = -c_1/y+b_2 \ln y+\Lambda_2(y). \tag{15b}$$

Here the Λ_n are given by

$$\Lambda_n(y) = U_n(\tfrac{1}{2})+2c_1\delta_{2n}+a_n(y-\tfrac{1}{2})+\int_{\frac{1}{2}}^{y} (T_n-\delta_{2n}c_1/y^2-b_n/y-a_n)\,dy \quad (n = 1, 2). \tag{16}$$

The integrals are bounded as $y \to 0$.

If we now match the mean velocity, we get

$$U_* \ln\epsilon = -k[(1+\epsilon D_2+\epsilon^2 D_4+\ldots)+U_*(D_1+\epsilon D_3+\ldots)], \tag{17}$$

where

$$k = 1/b_1,$$

$$D_1 = \Lambda_1(0)-\lambda_1(\infty), \quad D_2 = -kb_2,$$

$$D_3 = \Lambda_2(0)-\lambda_2(\infty)-kb_2[\Lambda_1(0)-\lambda_1(\infty)].$$

3. Skin friction at moderately large Reynolds number

The effects on the skin friction are readily seen by rewriting (17) in terms of the skin-friction coefficient $C_f = 2U_*^2$:

$$(2/C_f)^{\frac{1}{2}} = -k^{-1}\ln\epsilon+D_1-b_2\epsilon\ln\epsilon+(D_3+kb_2D_1)\epsilon+\ldots \tag{18}$$

and

$$\epsilon = [R(\tfrac{1}{2}C_f)^{\frac{1}{2}}]^{-1}. \tag{19}$$

Here the effect which could manifest itself first at moderately large Reynolds number is the dependence of the coefficient of the logarithmic term and the additive coefficient.

It is indeed well known that at moderately large Reynolds number there are significant departures from the classical logarithmic law. Patel & Head (1969) and others have shown that the skin friction in pipe flow falls below the logarithmic value at low Reynolds number; Patel & Head fitted their data to

$$C_f = 0{\cdot}079R^{-\frac{1}{4}} \tag{20}$$

in the range $3000 < Re < 10000$. We now examine whether the extended logarithmic law (18) can describe the skin-friction measurements in a pipe in the range for which the Blasius law (20) is commonly used.

We first need to determine the value of k. In the literature the values of k employed range from 0·36 to 0·417 (Schlichting 1968; Coles 1956, 1968; Goldstein 1965). Further, a closer examination of the available experiments shows that k is evaluated by fitting a straight line to the plot of $(2/C_f)^{\frac{1}{2}}$ *vs.* $\ln\epsilon$ over a wide range of Reynolds numbers. However, we see that according to (18) the curve tends to become a straight line asymptotically only as R approaches infinity. Thus in the present context the value of k has to be found by examining the slope of the curve $(2/C_f)^{\frac{1}{2}}$ *vs.* $\ln\epsilon$ at large Reynolds number rather than by fitting a straight line over a wide range of Reynolds numbers as was done by other authors. An asymptote was drawn to the $(2/C_f)^{\frac{1}{2}}$ *vs.* $\ln\epsilon$ curve (Schlichting 1968, figure 20.9, p. 573) and the appropriate value of k seems to be 0·398, which is rounded off to 0·4. This value is compatible with values used by other authors, e.g. Hinze (1964) and Patel & Head (1969), although latter authors used a slightly different value of 0·41.

Consider now the quantity Q ($1/\sqrt{2}$ times the difference between the left-hand side $(\frac{1}{2}C_f)^{-\frac{1}{2}}$ of (18) and the first two terms on right-hand side),

$$Q\sqrt{2} = (2/C_f)^{\frac{1}{2}} + k^{-1}\ln\epsilon - D_1 \tag{21}$$

$$= -b_2\epsilon\ln\epsilon + (D_3 + kb_2D_1)\,\epsilon + \dots, \tag{22}$$

which represents the departure from the classical logarithmic skin-friction law at moderately large Reynolds number. Patel & Head found that to the accuracy of their measurements the slope of the logarithmic velocity profile is independent of Reynolds number. If we accept this as at least a good engineering approximation, it follows that $b_2 = 0$ and (22), using (19), becomes

$$Q = D_3(R\sqrt{C_f})^{-1} + \dots. \tag{23}$$

If Q is plotted against $(R\sqrt{C_f})^{-1}$, the curve ought to approach a straight line as the Reynolds number approaches infinity. A plot of Q is given in figure 1 on the basis of data of Patel & Head for pipe flows. The slope of the asymptote gives $D_3 = 150$.

The extended logarithmic skin-friction law† is

$$C_f^{-1} = 4{\cdot}07\log_{10}(R\sqrt{C_f}) - 0{\cdot}93 + 150(R\sqrt{C_f})^{-1} + o(R/\sqrt{C_f})^{-1}. \tag{24}$$

† The extended logarithmic law can be obtained by extending (see N. Afzal 1973, NAL AE-TM-2-73) a simple argument due to Millikan (1938) and Gill (1968).

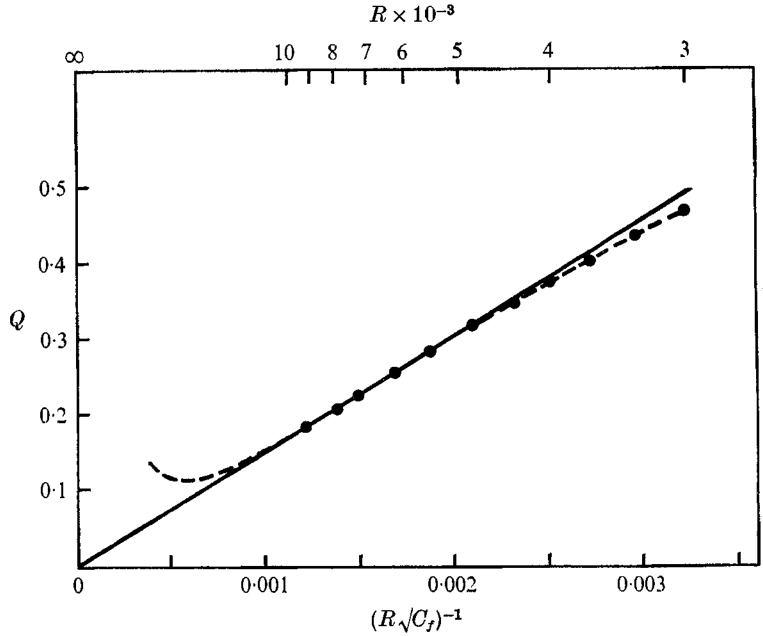

FIGURE 1. Departure Q, defined as $C_f^{-\frac{1}{2}} - 4{\cdot}07 \log_{10}(R\sqrt{C_f}) + 0{\cdot}93$, of skin friction from classical logarithmic law for pipe flows at moderately large Reynolds number. ●, Patel & Head; ——, $Q = 150/(R\sqrt{C_f})$; – – –, Blasius law.

The above law agrees to within 1 % with the data of Patel & Head (1969) and the Blasius law (20) down to a Reynolds number of 3000. Since the intermittency disappears at approximately this Reynolds number, this value may be taken to denote the upper limit of transition, and the range of applicability of law (24) may be taken as the entire turbulent regime after the completion of transition.

4. Velocity law in the overlap and sublayer regions

The velocity distribution in the overlap region is given by

$$\frac{U}{U_*} \sim \frac{1}{k}\ln\eta + \lambda_1(\infty) - \frac{C_1}{\eta} + \ldots + \epsilon\left(a_2\eta + b_2\ln\eta + \lambda_2(\infty) - \frac{C_2}{\eta} + \ldots\right) + \ldots \quad \text{as} \quad \eta \to \infty, \tag{25}$$

$$\frac{1-U}{U_*} \sim -\frac{1}{k}\ln y - \Lambda_1(0) - a_2 y + \ldots + \epsilon\left(\frac{C_1}{y} - b_2\ln y - \Lambda_2(0) - a_3 y + \ldots\right) + \ldots \quad \text{as} \quad y \to 0, \tag{26}$$

and the corresponding relations for Reynolds stress can also be readily written down. As mentioned earlier, from the observations of Patel & Head, $b_2 = 0$. Now from (25) and (26) one notices that the higher order effects influence the additive coefficients of the laws. This is confirmed by the experiments of Patel & Head for pipe as well as for channel flows. If we consider the quantity

$$B = \lambda_1(\infty) + \epsilon(\lambda_2(\infty) + a_2), \tag{27}$$

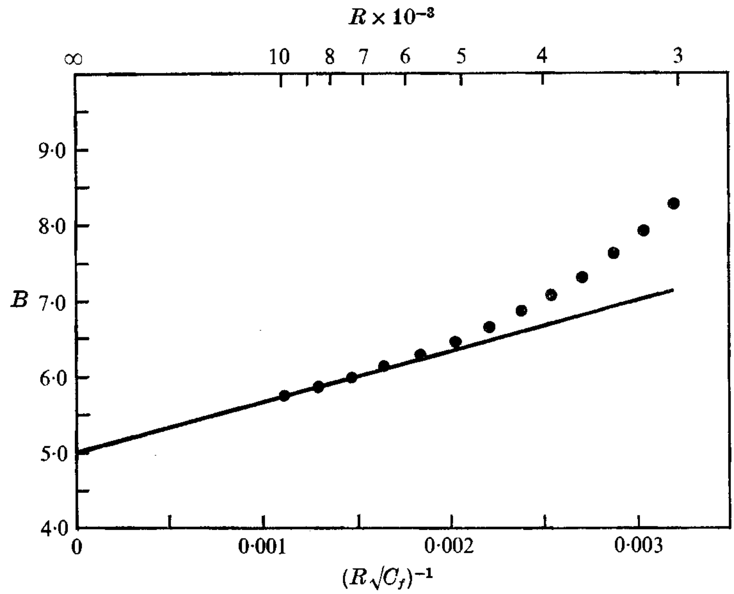

FIGURE 2. Variation of B (the additive terms in the logarithmic velocity law for law of the wall) for pipe flows at moderately large Reynolds number. ●, Patel & Head; ——, $B = 5{\cdot}0 + 667/(R\sqrt{C_f})$.

the additive terms in the overlap region in the wall co-ordinates (25), the higher order effect is of the order ϵ or $(R\sqrt{C_f})^{-1}$. The additive coefficient B for pipe flows is plotted against $(R\sqrt{C_f})^{-1}$ in figure 2 from the data of Patel & Head, and the asymptote to the data is found to be

$$B = 5 + 667(R\sqrt{C_f})^{-1} + \dots. \tag{28}$$

As R approaches infinity the additive coefficient thus approaches 5, which is the value recommended by Coles (1968). At lower Reynolds number the asymptote (28) underestimates the coefficient, by about 5 % at $R = 4000$ and about 12 % at $R = 3000$, when compared with the data of Patel & Head for pipe flows.

The velocity distribution in the sublayer is given by

$$U = U_*[\eta + \tfrac{1}{24}\eta^4\tau_1'''(0) + o(\eta^4)] + U_*\epsilon[-\tfrac{1}{2}\eta^2 + \tfrac{1}{24}\eta^4\tau_2'''(0) + o(\eta^4)] + o(U_*\epsilon). \tag{29}$$

In (29) the term of order U_* corresponds to the usual law of the sublayer (see Rotta 1962, p. 59) and the higher order term is the correction for lower Reynolds number. Also, the total shear stress in the inner layer behaves like $1 - 2\epsilon\eta$.

5. Measurement of skin friction at moderately large Reynolds number

Wall similarity forms the basis for many skin-friction measuring devices including Presten tube, Stanton tube and the hot-film gauge. The range of these devices can be extended by using the generalization of the wall law given by (5). For instance, the relationship between the Preston-tube variables can be modified using the correspondence with the extended wall law

$$\tau_0 D^2/4\rho\nu^2 = f_1 + f_2\epsilon + f_3\epsilon^2 + \dots, \tag{30}$$

where the f_n are functions of $(p_0-p)D^2/4\rho\nu^2$. Here the usual notation is used. Alternative forms, as suggested by Head & Ram (1971), can also be used. The above relation thus provides a way of including moderate Reynolds number effects in Preston-tube calibration. However, calibration of these higher order effects carried out in, say, pipe flow can be applied to boundary-layer measurements only if the higher order effects have the same structure in both the flows.

6. Conclusion

(i) It is possible to analyse properties of turbulent shear flows at moderately large Reynolds number by using the method of matched asymptotic expansions. It is not necessary to invoke any closure hypotheses for obtaining the functional forms of asymptotic laws for the skin friction and velocity profile.

(ii) The extended skin-friction law (24) is different from the classical logarithmic law in that the additive coefficient depends on Reynolds number The extended law accurately predicts the skin friction over the entire turbulent regime $R > 3000$ (the upper limit of the transition regime). Before, two empirical laws were needed for this range.

(iii) The additive coefficients in the overlap velocity laws (25) and (26) have a similar Reynolds number dependence.

(iv) The extended form of the law of the wall provides a basis for extending the range of skin-friction and heat-transfer measuring techniques based on wall similarity.

(v) The close agreement of results of the present work with the experimental data strongly supports the nature of higher order effects discussed here. However, the values of k, D_1, D_2, D_3, etc., may need slight modifications when more accurate data becomes available.

(vi) Comparison with channel flows has not been attempted as the experimental results of Patel & Head (1969), Clark (1968) and Beavers, Sparrow & Lloyd (1971) are controversial, and hence, no definite trends can be established. For example, the data of Patel & Head (1969) show that the additive term in the law of the wall remains constant (equal to the universal value) for all Reynolds numbers while Clark (1968) shows an increase in the value of the additive term. On the other hand, the skin-friction measurement of Beavers *et al.* (1971) also shows a large departure from those of Patel & Head (1969). Thus more extensive and accurate experiments are needed in channel flows.

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REFERENCES

AFZAL, N. 1971 Compressible turbulent boundary layer at moderately large Reynolds number. *Dept. of Aeron. Engng, Ind. Inst. Tech., Kanpur, Tech. Rep.* no. 16/1971.

AFZAL, N. 1973 *J. Fluid Mech.* **57**, 1.

AFZAL, N. & YAJNIK, K. S. 1971 *Z. angew. Math. Phys.* **22**, 899.

BEAVERS, G. S., SPARROW, E. M. & LLOYD, J. R. 1971 *J. Basic Engng*, **93**, 296.

CHAWLA, T. C. & TENNEKES, H. 1973 *Int. J. Engng Sci.* **11**, 45.

CLARK, J. A. 1968 *J. Basic Engng*, **90**, 445.

COLES, D. 1956 *J. Fluid Mech.* **1**, 191.

COLES, D. 1968 *Proc.* 1968 *AFOSR-IFP-Stanford Conf. on Computation of Turbulent Boundary-layers*, vol. 2, pp. 1–45.

GILL, A. E. 1968 *J. Math. Phys.* **47**, 437.

GOLDSTEIN, S. 1965 *Modern Developments in Fluid Dynamics*, p. 366. Dover.

HEAD, M. R. & RAM, V. V. 1971 *Aero. Quart.* **22**, 295.

HINZE, J. O. 1959 *Turbulence*, p. 515. McGraw-Hill.

HINZE, J. O. 1964 In *Mechanics of Turbulence* (ed. A. Favre). Gordon & Breach.

HUFFMANN, G. D. & BRADSHAW, P. 1972 *J. Fluid Mech.* **53**, 45.

MELLOR, G. L. 1972 *Int. J. Engng Sci.* **10**, 851.

MILLIKAN, C. B. 1938 *Proc. 5th Int. Congr. Appl. Mech.*, pp. 386–392.

PATEL, V. C. & HEAD, M. R. 1969 *J. Fluid Mech.* **38**, 181.

ROTTA, J. C. 1962 In *Progress in Aeronautical Sciences*, vol. 2. Pergamon.

SCHLICHTING, H. 1968 *Boundary-Layer Theory*, 6th edn. McGraw-Hill.

TENNEKES, H. 1966 *Appl. Sci. Res.* **15**, 232.

TENNEKES, H. 1968 *A.I.A.A. J.* **6**, 1735.

VAN DYKE, M. 1964 *Perturbation Methods in Fluid Mechanics*. Academic.

YAJNIK, K. S. 1970 *J. Fluid Mech.* **42**, 411.